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LETTER TO THE EDITOR

Comments on 'Upper bound solution to elasticity problems: A unique property of the linearly conforming point interpolation method (LC-PIM)'*

by G. R. Liu and G. Y. Zhang, *International Journal for Numerical Methods in Engineering*, DOI: 10.1002/nme.2204 [1]

From: *G. R. Liu*[1,2] *and Guiyong Zhang*[2]
[1]*Centre for Advanced Computations in Engineering Science* (*ACES*),
Department of Mechanical Engineering, *National University of Singapore*,
9 Engineering Drive 1, *117576*, *Singapore*
[2]*Singapore-MIT Alliance* (*SMA*), *E4-04-10*, *4 Engineering Drive 3*, *117576*, *Singapore*

The authors appreciate very much the comments and questions from the reader [2]. We will try our best to address those comments and questions, and also make use of this opportunity to further clarify a few issues.

1. ON THE UNIQUENESS OF PROPERTY OF THE METHOD

First, we have re-checked again on our code that produced Figure 4, and could not found any errors or mistakes. The results plotted in Figure 4 are correct. We have also derived the formulation analytically (which can be done for 1D problems), and confirmed the correctness of the solutions.

In trying to understand the point of the comments on the uniqueness of the method, we read again the statement we made at the beginning of Section 5.2.2 in our paper. Yes, we did claim that 'the solution of LC-PIM gives the upper bound of the exact solution in energy norm'. The full sentence is '… except for a few trivial cases, the solution of LC-PIM gives the upper bound of the exact solution in energy norm'. We made it clear that there are exceptions of 'a few trivial cases'. The point in Figure 4 shows one of such trivial cases. To further clarify these trivial cases, we went much further to write a lengthy Section 5.2.3 to elaborate intensively on these trivial cases.

We believe that the upper bound property of LC-PIM is unique in the following ways.

- It achieves an upper bound by a unique way: properly arranged *node-based* smoothing operation. Compared with the traditional ways such as the equilibrium model, it is an entirely different approach and thinking toward obtaining an upper bound solution. It is a very simple and practical way of introducing the softness into the model leading to an upper bound solution. It is because of this uniqueness, there will be exceptional trivial cases when the smoothing effects

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are too small to introduce sufficient softness to the model that uses very few element/cells (see Section 5.2.3).

- It always achieves an upper bound using a *reasonably fine* mesh that can be the same as the finite element method (FEM) mesh without any change of settings in the original problem and the boundary conditions, and hence the method is very practically routinely workable for all complex problems of elasticity whenever an FEM model can be built.
- It can be considered as a quasi-equilibrium model, as it has some properties of the equilibrium model. It gives up on seeking equilibrium for every point in the entire domain, but achieves equilibrium status only in node-based cells resulting in sufficient softening effects. The LC-PIM gives up a theoretically 'rigorous' upper bound, and treats for a useful and easily obtainable upper bound for practical problems.

By a reasonably fine mesh, we mean two things. First, the number of cells/elements should not be too small so that we can have sufficient number of nodes for the node-based smoothing operations to take effect. Second, the FEM solution based on such a mesh should not be too far away from the exact solution. For the 1D force-driven problem, we found that the minimum number of elements (evenly discretized) is two: as long as one uses more than two elements for this problem, an upper bound solution can *always* be obtained for this problem. Note that the minimum number of elements also depends on the element/cell mesh and the division of the smoothing domain as shown in Figure 4: when one uses two elements, but deliberately reduces too much the size of the smoothing domain for the middle node, one may not be able to obtain the upper bound solution.

As the background of the authors is in engineering, we often look for practical methods for a desired solution that may not be workable for specially designed cases. The upper bound solution of LC-PIM is with conditions, but it is very conveniently obtainable, very useful practically and always workable as long as we do not use very coarse mesh, shown in all the example problems of 1D, 2D and 3D presented in the paper.

Note that we did not mean that the LC-PIM is *the only* method that can produce an upper bound solution. There are other methods capable of producing upper bound solution for simple problems. For example, the so-called force methods or the equilibrium models mentioned in the comments can be very good ones. These methods have a long history, and are used by many. We will not comment further on this, as we do not have sufficient experience on the details of these methods. We are now trying to learn more about these methods. All the authors can say now is that the LC-PIM can have a very good chance to be used in practical applications, as it needs very little change to the existing FEM codes. As long as a reasonably fine model (non-trivial) for an FEM solution can be built for the problem, an upper bound solution can always be produced using LC-PIM.

2. ON THEOREM 3 AND RIGOROUS PROOF OF THE UPPER BOUND FOR GENERAL CASES

As we stated briefly in the paper, our search for a practical method for bounding a solution to general problems from the above was very long, and was not successful until we found this unique property of LC-PIM. We got very excited when we first discovered this property, but for long time we could not prove it. We then went to search for counter examples, and found these trivial cases. Therefore, a rigorous proof would not be an easy task, as there are clearly counter examples and

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hence it requires a *precise* definition on what constitutes ‘trivial cases’. We went further to find out what is the next possible thing that we can do to reveal more in depth information for this bounding property of LC-PIM, which leads to Theorem 3. At first glance, it may seem useless: if a basis with exact solution can be found, FEM will simply reproduce the exact solution and hence there is no need for any other forms of solutions. However, this theorem is quite important to give some important support to the fact that when the mesh size becomes small; the LC-PIM can *always* produce an upper bound solution! Together with Theorem 2, Theorem 3 gives the foundation for our argument of ‘battle between stiffening and softening’ detailed in Section 5.2.3, which may be a ‘trace’ of a proof of the upper bound property of LC-PIM for general cases when the exact solution cannot be included into the basis. It is not a rigorous proof, but it provides an in-depth *intuitive* explanation on why LC-PIM can always produce an upper bound solution when a reasonably fine mesh is used. The essential point of the argument is that the softening effect provided by the LC-PIM is always larger than the stiffening effect induced by the FEM model, when a reasonably fine mesh is used. Therefore, when the FEM model ‘drags’ the solution down *a little* from the exact one, the LC-PIM model ‘pushes’ the solution up *more* from the exact solution. The ‘more-pushing-up’ fails for these trivial cases when the number of the elements/cells used is too small. Why the LC-PIM model pushes up more than the dragging down of the FEM model? This is because the FEM model underestimates the strain energy by approximating the *continuous* exact strain field by a piecewise-constant strain field over the elements (consistency reduced a little *within* the elements), but the LC-PIM model overestimates the strain energy by approximating the *discontinuous* FEM strain field by a piecewise-constant strain field over the node-based smoothing cells (consistency increased a lot within the cells). Therefore, the overestimation will be larger than the underestimation as long as the mesh used is reasonably fine. The detailed discussion on this is given in Section 5.2.3, in lieu of a rigorous proof that we could not provide.

We understand that the argument of ‘battle between stiffening and softening’ is not a rigorous proof. To conduct a rigorous proof, one may need to define clearly how fine a mesh is ‘reasonably fine’ to ensure an upper bound. This is up to now still an open question, but we feel that some kind of more rigorous proof can be done following the above-mentioned arguments, and we are still working on this. Our numerical test on the force-driven 1D problem shows that the minimum number of elements is two. In practice, we engineer to use a lot more than two elements in solving practical problems.

3. ON FORCE-DRIVEN AND DISPLACEMENT-DRIVEN PROBLEMS

In this paper, we only discussed the so-called force-driven problems, which was intentionally done to avoid possible confusions that can defocus the central topic of this paper. For displacement-driven problems, the revisal is expected: the LC-PIM will produce a lower bound, and the corresponding compatible model (such as the FEM model) produces, on the other hand, an upper bound. Therefore, we still bound the solution from both sides using one same model. This should hold also for mixed problems, but we do not want to make it a point at this stage, before conducting detailed studies.

4. ON ZERO ENERGY MODES

Question: Isn’t it possible to have, for some particular geometries, zero-energy modes?

Answer: In our past experience in engineering, we could not yet find any particular geometry of structure that can be well posed (statically), and yet the LC-PIM model will become singular, as analyzed in the last paragraph of Section 5.1. The point is that the LC-PIM modeling procedure itself will not introduce additional singularity to the system.

Note that being free from zero-energy modes ensures a unique *numerical* solution for static problems. However, there can be spurious modes at the higher energy level for a method that has too much smoothing effects, such as the LC-PIM. This kind of phenomenon is quite similar to that found in an equilibrium model. This kind of spurious modes will appear when free vibration analysis is performed using such a method. It can also show as an instability behavior when a transient analysis is conducted. We are currently investigating this and hopefully we can report the detailed findings in the near future with possible remedies to suppress the spurious modes. One of such a remedy is the recently formulated ES-FEM model [3].

5. ON QUASI-EQUILIBRIUM MODEL

Question: The constant strain triangle satisfies the conditions that are invoked for the LC-PIM solution in Remark 9. The authors consider it a quasi-equilibrium model?

Answer: Very good point! The answer to the question is ‘no’. If only the condition on equilibrium mentioned in Remark 9 is met in a model, we should not consider it a quasi-equilibrium model. The model has to be not fully compatible. The LC-PIM can be considered as a quasi-equilibrium model because it is not full compatible: the displacement is compatible *only* on the interfaces of the smoothing domains, as mentioned in Remark 9. It is not compatible at any point in the problem (except on these interfaces), in terms of displacement-strain relations. The condition on equilibrium along given Remark 9 is not sufficient, as an FEM model of constant triangular elements is clearly an example: it can also meet the equilibrium condition in Remark 9, but is a fully compatible model in the entire problem domain. Note also that the LC-PIM has also additional properties given in Remarks 11 and 13. We have found that LC-PIM is free of volume-locking, which is another typical behavior of the equilibrium model. We have not discussed this at all in this paper. Moreover, in an LC-PIM model, there can be spurious modes at the higher energy level, which is another typical phenomenon similar to that found in an equilibrium model (see also item 4). This is also not discussed in this paper. For a good discussion on the quasi-equilibrium feature of LC-PIM, a much more complete study should be conducted, comparing all the typical features of both compatible and equilibrium models, which was not the focus of this paper. This kind of discussion can be very useful, as it can lead to better models with desired properties.

6. OTHER MATTERS

We should emphasize again that LC-PIM is not a magic method of perfection for all the situations, but one with special and important properties. We are still searching for a rigorous proof or a set of criteria that ensures an upper bound, so as to remove the worries of many. This will not be easy, but we believe that it is possible. We hope people from the mathematical community can help us on this, as it not an easy task for engineers like the authors to complete it rigorously.

Finally, we would like to add our experience on the LC-PIM. The LC-PIM is a very practical, robust and very convenient way to provide an upper bound. We were very excited when we

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discovered this property of LC-PIM, and we hope readers interested in this topic can find the same, we can provide the basic source codes upon request, so that interested readers can try it out by themselves. We hope the confidence on the method will be firmly established for the upper bound solutions for practical problems. Hopefully, commercial software developers can add this upper bound function in their codes and make it available to analysts in the area of structural design. We also hope that the idea of introducing smoothing operation into numerical models can be explored further, because there are many ways of smoothing operation and different means to introduce the smoothing operations. The authors believe that it is a promising direction of establishing outstanding numerical methods/techniques for desired properties and for different types of problems.

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